§6 Curvature

We continue the discussion under which conditions there are horizontal sections

$$s : U \longrightarrow L^\times$$

in a line bundle $L \to M$ with connection ∇ over an open subset U of M.

Assume, that U is a coordinate neighbourhood with coordinates $q^1, \dots q^n$, i.e. we have a diffeomorphism $\varphi = (q^1, \dots q^n) : U \longrightarrow V \subset \mathbb{R}^n$. At a given point $a \in U$ we find a horizontal lift $\hat{\gamma}_1$ of the curve $\gamma_1(t) = \varphi^{-1}(\varphi(a) + te_1)$ representing $\frac{\partial}{\partial q^1}$, $t \in I_1$. We set $s(\gamma_1(t)) := \hat{\gamma}_1(t)$, $t \in I_1$. For each $t \in I_1$ the curve $\gamma_2^t(u) := \varphi^{-1}(\varphi(a) + te_1 + ue_2)$, $u \in I_2 \subset \mathbb{R}$, has again a horizontal lift $\hat{\gamma}_2^t(u)$ through $\hat{\gamma}_1(t)$. We set $s(\gamma_2^t(u)) := \hat{\gamma}_2^t(u)$. In the same way one can proceed with $3, \dots, n$. But let us stick to the case $n = 2$. Then s as above defines a section $s : U \to L^\times$. But is s horizontal? s is horizontal if $\nabla_X s = 0$, and this condition is satisfied if for $\nabla_1 = \nabla_{\frac{\partial}{\partial q^1}}$ and $\nabla_2 = \nabla_{\frac{\partial}{\partial q^2}}$ the conditions $\nabla_i s = 0$ are satisfied, $i = 1, 2$. Now, $\nabla_2 s = 0$ is evident by definition, since $u \mapsto s(\gamma_2^t(u)) = \hat{\gamma}_2^t(u)$ is horizontal for each $t \in I$.

If now ∇_1 and ∇_2 commute, we have $\nabla_2 \nabla_1 s = \nabla_1 \nabla_2 s = 0$, (because of $\nabla_2 s = 0$) and it follows that (for fixed t)

$$\eta(u) = \nabla_1 s(t,u)$$

is a horizontal lift of $\gamma_2(t,u)$ with $\eta(0) = 0$ ($\eta(0) = \nabla_1 \hat{\gamma}_1(t) = 0$ since $\hat{\gamma}_1$ is a horizontal lift of γ_1). Eventually, since the horizontal lift is unique, we have

$$\nabla_1 s(t,u) = \eta(u) = 0 .$$

We have shown "$3° \Rightarrow 2°$" of the following result:

(6.1) PROPOSITION: For a line bundle $L \xrightarrow{\pi} M$ with connection ∇ the following properties are equivalent:

1° Parallel transport is locally independent of the curves.

2° Every point $a \in M$ has an open neighbourhood $U \subset M$ with a horizontal section $s \in \Gamma(U,L)$, $s \neq 0$.

3° $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = 0$ for $X, Y \in \mathcal{V}(U)$, $U \subset M$ open.

□ Proof. "1° <=> 2°" is the content of (5.7), and "2° => 3°" is similar to the considerations before the proposition. □

(6.2) DEFINITION: For a line bundle $L \xrightarrow{\pi} M$ with connection ∇:

1° The CURVATURE operator is

$$F = F_\nabla : \mathcal{V}(M) \times \mathcal{V}(M) \longrightarrow \mathcal{V}(M)$$
$$(X,Y) \longmapsto \frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)$$

2° The CURVATURE form on $L^\times$ is $d\alpha \in \Omega^2(L^\times)$ where $\alpha \in \Omega^1(L^\times)$ is the connection form on $L^\times$ (cf. (4.5)).

3° The CURVATURE form $\Omega = \mathrm{Curv}(L,\nabla) \in \Omega^2(M)$ is defined as follows: If $(U_j)_{j\in I}$ is an open cover of M with trivializations $\varphi_j : L_{U_j} \to U_j \times \mathbb{C}$ and local gauge potentials $\alpha_j \in \Omega^1(U_j)$ for ∇ then

$$\Omega|_{U_j} := d\alpha_j \,, \quad j \in I.$$

The last expression is well-defined, since we know from (4.4)

$$[Z] \quad \alpha_k - \alpha_j = \frac{1}{2\pi i}\, dg_{jk}\, g_{jk}^{-1} \quad \text{on } U_{jk} = U_j \cap U_k .$$

(6.3) <u>PROPOSITION:</u> For a connection ∇ on a line bundle we have:

1° $F_\nabla(X,Y) = \Omega(X,Y)$ for $X,Y \in \mathcal{V}(M)$

2° $\pi^*\Omega = d\alpha$

3° $s^* d\alpha = \Omega|_U$ for any section $s \in \Gamma(U, L^\times)$

□ <u>Proof</u>. 1° It is easy to show that F_∇ is bilinear over $\mathcal{E}(M)$ and therefore is a 2form. But 1° asserts more, namely that this 2form is Ω. This can be checked by showing it over each U_j, i.e. we need to show it only for trivial bundles with $\nabla_X f s_1 = (L_X f + 2\pi i\, \alpha(X) f)\, s_1$, $s_1(a) = (a,1)$ and $s = f s_1 = (a, f(a))$, where $\alpha \in \Omega^1(U_j)$ is the local gauge potential (not the global one in 2°):

$$\begin{aligned}
[\nabla_X, \nabla_Y] f s_1 &= \big(L_X L_Y f - L_Y L_X f + 2\pi i\,(\alpha(X) L_Y f - \alpha(Y) L_X f) \\
&\qquad + 2\pi i\, L_X(\alpha(Y) f) - 2\pi i\, L_Y(\alpha(X) f)\big)\, s_1 \\
&= \big(L_{[X,Y]} f + 2\pi i\,(L_X \alpha(Y) - L_Y \alpha(X)) f\big)\, s_1
\end{aligned}$$

$$\nabla_{[X,Y]} f s_1 = \left(L_{[X,Y]} f + 2\pi i \, \alpha[X,Y] f \right) s_1$$

Therefore,

$$\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \right) f s_1 = 2\pi i \Big(\overbrace{L_X \alpha(Y) - L_Y \alpha(X) - \alpha([X,Y])}^{d\alpha(X,Y)} \Big) f s_1,$$

i.e. $\quad F_\nabla(X,Y) s = d\alpha(X,Y) s = \Omega(X,Y) s$

$$\Rightarrow \quad F_\nabla = \Omega .$$

2° From section 4 we know :

$$\alpha|_{L^\times_{U_j}} = \pi^* \alpha_j + \frac{1}{2\pi i} \varphi_j^* \left(\frac{dz}{z} \right), \quad j \in I,$$

hence,

$$d\alpha|_{L^\times_{U_j}} = \pi^* d\alpha_j = \pi^*(\Omega|_{U_j}), \; j \in I,$$

3° The same relation between α and α_j yields for $s \in \Gamma(U, L)$

$$s^*\alpha = s^* \pi^* \alpha_j + \frac{1}{2\pi i} s^* \varphi_j^* \left(\frac{dz}{z} \right) = \alpha_j + \frac{1}{2\pi i} (\varphi_j \circ s)^* \left(\frac{dz}{z} \right) \quad \text{on } U \cap U_j$$

and

$$s^* d\alpha = d\alpha_j = \Omega \quad \text{on } U \cap U_j . \qquad \square$$

[2.12.09]

In the following we want to show how the parallel transport can be expressed by a suitable integral over the curvature form $\Omega = \mathrm{Curv}(L, \nabla)$.

Let $\mathcal{L}(a)$ be the set of all loops (closed smooth curves) which start and end in a fixed point $a \in M$. Then the parallel transport

$$\mathbb{P}^{\gamma}_{t_0, t_1} : L_a \to L_a \;, \quad \gamma(t_0) = \gamma(t_1) = a \,,$$

is determined by a complex number $Q(\gamma) \in \mathbb{C}^\times$:

$$\mathbb{P}^\gamma_{t_0,t_1} = \mathbb{P}^\gamma : \ell \mapsto Q(\gamma)\ell \ , \ell \in L_a .$$

(6.4) <u>**PROPOSITION**</u>: Let $S \subset M$ be an oriented compact surface in M with boundary ∂S parametrized by $\gamma \in \mathcal{L}(a)$. The parallel transport $\mathbb{P}^\gamma : L_a \to L_a$ along γ is given by

$$Q(\gamma) = \exp\left(-2\pi i \int_S \Omega\right) , \text{ i.e.}$$

$$\mathbb{P}^\gamma(\ell) = Q(\gamma)\,\ell \ , \ \ell \in L_a .$$

□ <u>Proof</u>. It is enough to show the result locally, hence we can assume the line bundle to be trivial: $L = M \times \mathbb{C}$. The horizontal lift of $\gamma(t) \in M$ has the form

$$\hat{\gamma}(t) = (\gamma(t), \xi(t)) \ , \quad t \in [t_0, t_1] = J,$$

with $\xi(t) = \xi(t_0)\,\varrho(t) \in \mathbb{C}$, where

$$\varrho(t) = \exp\left(-2\pi i \int_{t_0}^{t} \alpha_j(\dot{\gamma}(s))\,ds\right)$$

because of

$$\dot{\xi} + 2\pi i\, \alpha_j(\dot{\gamma})\,\xi = 0$$

The integral is

$$\int_{t_0}^{t_1} \alpha_j(\dot{\gamma}(s))ds = \int_\gamma \alpha_j = \int_{\partial S} \alpha_j = \int_S d\alpha_j = \int_S \Omega \ \text{(Stokes)}.$$

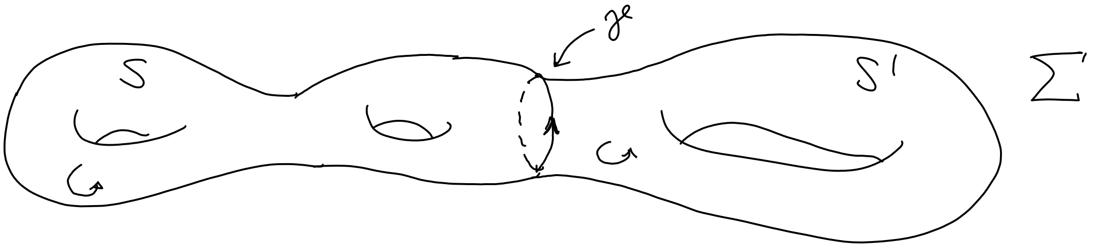
Now $\mathbb{P}^\gamma(a,z) = (a, z\varrho(t_1)) = \varrho(t_1)(a,z) = Q(\gamma)(a,z)$, where $z = \xi(t_0)$, $\ell = (a,z) \in L_a$. And

$$Q(\gamma) = \varrho(t_1) = \exp\left(-2\pi i \int_S \Omega\right).$$ □

Instead of restricting to smooth curves one often uses the more general class of piecewise smooth curves with the same results for lifting to horizontal curves and for parallel transport.

The last result leads to an integrality condition for the curvature $\Omega = \mathrm{Curv}(L, V) \in \Omega^2(M)$ which is of a topological nature but which is also very interesting from the point of view of quantization. Let us explain this in some detail:

Let $\Sigma \subset M$ be an oriented, compact surface smoothly embedded into M. Assume that Σ is closed, i.e. Σ has empty boundary. Then Σ is a 2-dimensional oriented and compact submanifold of M. We can find a simple closed smooth curve γ dividing Σ into two parts S, S' such that S is an oriented compact surface with boundary ∂S parametrized by γ, S' is another oriented compact surface with boundary $\partial S'$ parametrized by γ^-, and $S \cup S' = \Sigma$, $S \cap S' = \partial S = \partial S'$ (as sets without orientation). For example:



Cutting the closed surface Σ into S and S'.

Let $a \in \partial S$ be the initial and end point of γ. Then the parallel transport along γ is given by

$$Q = \exp\left(-2\pi i \int_{\gamma} \alpha_j\right) = \exp\left(-2\pi i \int_S \Omega\right)$$

and the parallel transport along γ^- (which is γ with the opposite orientation: $\gamma^-(t) := \gamma(t_1 - t - t_0)$) is given correspondingly by

$$Q^- = \exp\left(-2\pi i \int_{\gamma^-} \alpha_j\right) = \exp\left(-2\pi i \int_{S'} \Omega\right).$$

This is true if Σ is contained in an open U_j where we have a trivialization $\varphi_j : L_{U_j} \to U_j \times \mathbb{C}$ with the local gauge potential α_j. The formulas

$$Q = \exp\left(-2\pi i \int_S \Omega\right), \quad Q^- = \exp\left(-2\pi i \int_{S'} \Omega\right)$$

hold true in general by cutting Σ into pieces which are in suitable U_j's.
Since Q^- is the inverse of Q we have

$$\begin{aligned} 1 = Q^- Q &= \exp\left(-2\pi i \int_{S'} \Omega\right) \exp\left(-2\pi i \int_S \Omega\right) \\ &= \exp\left(-2\pi i \left(\int_S \Omega + \int_{S'} \Omega\right)\right) = \exp\left(-2\pi i \int_\Sigma \Omega\right). \end{aligned}$$

As a consequence, $\int_\Sigma \Omega \in \mathbb{Z}$, which is the integrality condition.

(6.5) <u>PROPOSITION:</u> Let (L, ∇) be a line bundle with connection. Then the curvature $\Omega = \mathrm{Curv}(L, \nabla)$ satisfies the following integrality condition:

[G1] $\int_\Sigma \Omega \in \mathbb{Z}$ for every oriented closed compact surface $\Sigma \subset M$.

(6.6) **PROPOSITION:** A closed two form $\Omega \in \Omega^2(M)$ on a manifold M satisfies [G1] if and only if

[G2] The deRham cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ is in the image of $H^2(M, \mathbb{Z}) \xrightarrow{i^2} H^2(M, \mathbb{C})$.

Here the homomorphism i^2 is induced as part of the long exact sequence coming from the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 1,$$

where $\exp(z) = e^{2\pi i z}$, $z \in \mathbb{C}$.

In more concrete terms, [G2] is – via Čech cohomology – equivalent to

[G3] There exists an open cover $(U_j)_{j\in I}$ of M such that the class $[\Omega] \in H^2((U_j)_{j\in I}, \mathbb{C})$ contains a cocycle $c = (c_{ijk})$, with $c_{ijk} \in \mathbb{Z}$ for all $i, j, k \in I$ such that $U_{ijk} \neq \emptyset$.

We don't prove the equivalence of [G1] – [G3] which is a purely topological result attributed to A. Weil. We come back to the integrality conditions in section 9 on prequantization.